



TITLE:

The statement AN is equivalent to the statement  $n \leq \beta \omega \backslash \omega > c$  (Logic and the Foundations of Mathematics)

AUTHOR(S):

Kamo, Shizuo

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The statement AN is equivalent to the statement  $n(\beta\omega \setminus \omega) > c$

Shizuo Kamo (Univ. of Osaka Prefecture)

1. Introduction and results. A filter  $\mathcal{F}$  on  $\omega$  is said to be ample, if there is an infinite subset  $a$  of  $\omega$  such that, whenever  $x \in \mathcal{F}$ ,  $a \setminus x$  is finite. A filter  $\mathcal{F}$  on  $\omega$  is said to be weakly ample, if for each free ultrafilter (uf)  $\mathcal{U}$  on  $\omega$ , there is a function  $f$  on  $\omega$  such that  $f(\mathcal{U}) \supset \mathcal{F}$ . Let us denote by AN the statement: "every free weakly ample filter on  $\omega$  is ample." In [2], we showed

PROPOSITION 1.

- (i) AN implies the existence of  $c^+$  Ramsey ufs on  $\omega$ , where  $c$  denotes the cardinality of  $2^\omega$ .
- (ii) The existence of  $c^+$  Ramsey ufs on  $\omega$  does not imply AN.
- (iii)  $\textcircled{P}$  implies AN, where  $\textcircled{P}$  denotes the statement: "every free filter on  $\omega$  generated by a set cardinality less than  $c$  is ample."

It seems to be interesting to consider how strong the statement AN is. As to this, we first show

THEOREM 1. AN is equivalent to the statement that  $\beta\omega \setminus \omega$  can not be covered by a family of  $c$  nowhere dense sets, where  $\beta\omega$  denotes the Čech-Stone compactification of  $\omega$ .

Let us denote by  $n(\beta\omega \setminus \omega)$  the Baire number of  $\beta\omega \setminus \omega$  (i.e. the minimal cardinal of a family of nowhere dense sets covering  $\beta\omega \setminus \omega$ ). As to the Baire number of  $\beta\omega \setminus \omega$ , the systematic estimation was given and several consistencies were shown in [1]. In [1], it is shown

PROPOSITION 2 (5.2.V in [1]) The statement  $n(\beta\omega \setminus \omega) > c$  does not imply that  $\forall \kappa < c (|2^\kappa| \leq c)$ .

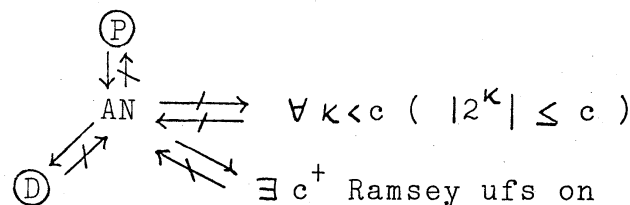
Since  $\textcircled{P}$  implies that  $\forall \kappa < c (|2^\kappa| \leq c)$  ([3]), by Theorem 1 and Proposition 2, AN does not imply  $\textcircled{P}$ .

Define the pseudo-ordering  $<^*$  on  $\omega^\omega$  by  $f <^* g$  iff  $\lim_{n \rightarrow \infty} (g(n) - f(n)) = \infty$ . A family  $F$  of subsets of  $\omega^\omega$  is said to be unbounded, if there does not exist  $g \in \omega^\omega$  such that, whenever  $f \in F$ ,  $f <^* g$ . Then, it holds

PROPOSITION 3 (4.6 and 4.7 in [1])

- (i) The statement  $n(\beta\omega \setminus \omega) > c$  implies the statement  $\textcircled{D}$ :  
"every unbounded family of subsets of  $\omega^\omega$  has the cardinality  $c$ ."
- (ii)  $\textcircled{D}$  does not imply that  $n(\beta\omega \setminus \omega) > c$ .

By Propositions 1~3 and Theorem 1, the following diagram holds.



The only interesting in the above diagram which is not mentioned is whether  $AN + \forall \kappa < c ( |2^\kappa| \leq c )$  implies  $\textcircled{P}$  or not. As to this, we show

THEOREM 2.  $AN + \forall \kappa < c ( |2^\kappa| \leq c )$  does not imply  $\textcircled{P}$ .

We shall prove Theorem 1 in the following section and Theorem 2 in section 3.

2. Proof of Theorem 1. We first show that  $n(\beta\omega \setminus \omega) > c$  implies AN. So, assume that  $n(\beta\omega \setminus \omega) > c$ . Let  $\mathcal{F}$  be any weakly ample filter on  $\omega$ . For each  $f \in \omega^\omega$ , set

$$D_f = \{ \mathcal{U} \in \beta\omega \setminus \omega ; f(\mathcal{U}) \supset \mathcal{F} \}.$$

Since  $\mathcal{F}$  is weakly ample, it holds that

$$\bigcup \{ D_f ; f \in \omega^\omega \} = \beta\omega \setminus \omega.$$

So, there is some  $f \in \omega^\omega$  such that  $D_f$  is not nowhere dense in  $\beta\omega \setminus \omega$ . Take an infinite subset  $a_0$  of  $\omega$  such that

$$(*) \quad \forall x \subset a_0 ( |x| = \omega \Rightarrow \exists \mathcal{U} \in D_f ( x \in \mathcal{U} ) ).$$

Set  $a_1 = f''a_0$ . Then, by (\*), it holds that  $a_1$  is infinite and  $\forall x \in \mathcal{F} ( a_1 \setminus x \text{ is finite} )$ . Hence,  $\mathcal{F}$  is ample.

Now, we shall prove that the inverse implication holds. The following fact which we shall use in the proof is well-known and easy.

FACT 1. There is a family  $W$  of subsets of  $\omega$  such that

- (1)  $|W| = c$ ,
- (2)  $\forall x \in W ( |x| = \omega )$ ,
- (3)  $\forall x, y \in W ( x \neq y \Rightarrow x \cap y \text{ is finite} )$ .

Assume that AN holds. Let  $\mathcal{Q} = \{D_\alpha; \alpha < c\}$  be any family of nowhere dense subsets of  $\beta\omega \setminus \omega$ . Let  $\langle a_\alpha \mid \alpha < c \rangle$  be a monotone enumeration of a family  $W$  of subsets of  $\omega$  which satisfies (1)  $\sim$  (3) in Fact 1. For each  $\alpha < c$ , take  $f_\alpha \in \omega^\omega$  such that  $f_\alpha : \omega \rightarrow a_\alpha$  one-to-one and onto. Define the filter  $\mathcal{F}$  on  $\omega$  by

$$x \in \mathcal{F} \text{ iff } \forall \alpha < c \forall \mathcal{U} \in D_\alpha (x \in f_\alpha(\mathcal{U})).$$

Then, it is easy to see that  $\mathcal{F}$  is free and not ample. So, by AN, there is  $\mathcal{U} \in \beta\omega \setminus \omega$  such that

$$\forall g \in \omega^\omega (g(\mathcal{U}) \not\supset \mathcal{F}).$$

Then, it holds that, for any  $\alpha < c$ ,  $\mathcal{U} \notin D_\alpha$ , since  $f_\alpha(\mathcal{U}) \not\supset \mathcal{F}$ . Hence,  $\mathcal{U} \notin \bigcup \mathcal{Q}$ . ■

3. Proof of Theorem 2. Let  $\mathcal{M}$  be a countable transitive model of ZFC + GCH. We shall show that a generic extension of  $\mathcal{M}$  on the poset  $P \times Q$  which will be defined below satisfies that  $\text{AN} + \forall \kappa < c (|2^\kappa| < c) + \neg \textcircled{P}$ . The poset  $P \times Q$  is alike the poset used in 5.V of [1]. Let  $P$  be the Solovay-Tennenbaum's poset used for the consistency of  $\text{MA} + |2^\omega| = \omega_2$ . Define the poset  $Q \in \mathcal{M}$  by, in  $\mathcal{M}$ ,

$$Q = \{q; \exists \alpha < \omega_1 (q : \alpha \rightarrow 2)\}.$$

Let  $G \times H$  be  $\mathcal{M}$ -generic on  $P \times Q$  and  $\tilde{\mathcal{M}} = \mathcal{M}[G \times H]$ . Then, similar arguments in [1] show that

$$\tilde{\mathcal{M}} \models "|2^\omega| = |2^{\omega_1}| = \omega_2 + \text{AN}."$$

We shall show that  $\tilde{\mathcal{M}} \models \neg \textcircled{P}$ . Since CH holds in  $\mathcal{M}$ , it holds that, in  $\mathcal{M}$ , there is a dense embedding from  $Q$  to  $P(\omega)/\text{finite}$ .

So, we may assume that  $H$  is  $\mathcal{M}$ -generic on  $(P(\omega)/\text{finite})^{\mathcal{M}}$ .

Define  $\mathcal{F} \in \tilde{\mathcal{M}}$  by

$$\tilde{\mathcal{M}} \models " \mathcal{F} = \{ x \subset \omega ; \exists a/\text{finite} \in H ( a \setminus x \text{ is finite } ) \}."$$

Since  $\tilde{\mathcal{M}} \models " |H| = \omega_1 "$ , it holds that

$$\tilde{\mathcal{M}} \models " \mathcal{F} \text{ is an } \omega_1\text{-generated free filter on } \omega."$$

Moreover, since  $H$  is not in  $\mathcal{M}[G]$ , we have that

$$\tilde{\mathcal{M}} \models " \mathcal{F} \text{ is not ample. }"$$

Hence,  $\tilde{\mathcal{M}} \models \neg \textcircled{P}$ .

■

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